Synchronization and Matched Filtering in Time-Frequency using the Sunflower Spiral

C. Willem Korevaar, André B.J. Kokkeleer, Pieter-Tjerk de Boer, Gerard J.M. Smit
Department of Electrical Engineering, Mathematics and Computer Science
University of Twente, The Netherlands
Email: c.w.korevaar@utwente.nl

Abstract—Synchronization and matched filtering of signals in time dispersive, frequency dispersive and time-frequency dispersive channels are addressed in this paper. The ‘eigenfunctions’ of these channels form the signal sets under investigation. While using channel-eigenfunctions is a first requirement for undistorted data transmission, a second necessity is to achieve good synchronization over the domains of time and frequency. The synchronization problem in time-frequency for non-stationary signals is discussed. A spiral correlation method is proposed to achieve synchronization and matched filtering in time-frequency. Spiral correlation, using the pattern of a sunflower, is simulated and evaluated. It is argued that partial spiral correlation can lead to a significant reduction in computational complexity necessary for synchronization. Generalizations and identities based on the fractional Fourier transform are provided which omit the need for fractional delay filters.

I. INTRODUCTION

An ideal communication system can be described as a transmitter and receiver which perfectly understand each other’s messages (share the same symbol set). They listen at the right time instance and wavelength (are synchronized). The environment does not distort the messages (signals are ‘eigenmessages’) and there is minimum signal degradation in noisy environments (signals are orthogonal and matched filtered). This paper shortly addresses all these aspects, but especially targets the synchronization and matched filtering problem of non-stationary signals with a time-frequency offset.

The eigenstructures of mobile radio channels have been discussed by a number of papers [1]–[3]. Usage of eigenfunctions of the channel ensures that the signals maintain their shape during transmission and are only attenuated/amplified by a complex factor, which eases equalization. Hermite functions are the approximate eigenfunctions of the linear time-variant (LTV) channel under the assumption of an elliptical scattering function [1].

Hermite functions form a set of orthogonal signals, are maximally concentrated in time and frequency (according to their second order moments over time and frequency [4]) and are eigenfunctions of the Fractional Fourier Transform (FrFT) [5]. Hermite functions are an interesting candidate as a signal basis for wireless communication systems, but also offer some key challenges like the time-frequency synchronization problem addressed in this paper.

Estimation of the time and frequency offset and synchronization for OFDM has received considerable research attention. Several techniques can be distinguished based on exploiting the cyclic prefix [6], adding pilot symbols [7] and analysis of cyclostationarities in the received signal [8]. Hermite functions are stretched over both time and frequency and are neither stationary over time nor over frequency. As a side note, higher-order Hermite functions tend to behave more and more like complex exponentials, which is known as the correspondence principle [9]. A cyclic extension cannot be used, due to the non-stationarity of Hermite functions. Accordingly, Hermite functions are sensitive to both time and frequency offsets, similar to the sensitivity of wavelet based transceivers to time and frequency offsets [10], [11]. Therefore, synchronization in time-frequency is essential.

This paper discusses the eigenfunctions for different channel models and pays attention to their respective synchronization problems to minimize inter-carrier interference (Section II and III). Optimal detection and matched filtering is generalized for arbitrary angles in time-frequency (Section IV). A spiral correlation method is proposed according to the sunflower spiral (Section V). Advantages are threefold. First, it is convenient to reformulate the 2-dimensional time-frequency offset problem as a function of one parameter. Second, using FrFT identities the usage of fractional delay filters can be omitted. Third, starting at the expected time-frequency offset offers the possibility to perform a partial spiral correlation until a ‘match’ is found. This may significantly reduce the number of computations. Simulations are performed for synchronization and matched filtering in time-frequency based on the sunflower spiral (Section VI). This paper concludes with a brief discussion and summary of the main results (Section VII).

II. EIGENSTRUCTURES OF WIRELESS CHANNELS

We describe a baseband multi-carrier transmit signal as an infinite sequence of symbols, each constructed by an ensemble of $K$ different subcarrier signals $s_k(t)$

$$x(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{K-1} A_{k,m} s_k(t - m \cdot T_s),$$  \hspace{1cm} (1)$$

where $A_{k,m}$ represents a real or complex value corresponding to the modulation scheme chosen. During each symbol time
$T_s$ a symbol with $K$ subcarriers is transmitted. In additive white Gaussian noise (AWGN) channels, orthogonality between the subcarriers $s_k$ and $s_{k′}$ and among the symbols is a requirement to recover the information $A_{k,m}$ without noise amplification.

Optimal signalling for stochastic wireless channels, in case of perfect channel knowledge, can be found by means of the Karhunen-Loève eigen-expansion of the channel (e.g. [12]). In practice due to a lack of perfect channel knowledge, computational complexity and rapidly changing channels, assumptions are made about the channel. We consider three different channel models based on linear time-invariant (LTI), linear frequency-invariant (LFI) and linear time-variant (LTV) channel models. To avoid distortion by the channel the signals $s_k$ (for any symbol) are chosen to be the eigenfunctions of the operator $\mathcal{H}$ resembling the channel

$$\mathcal{H}s_k = \lambda_k s_k,$$  \hspace{1cm} (2)

such that the shape of the subcarriers is preserved during transmission. The channel models yield an input-output relation, which is characterized by the Delay-Doppler spread function $S_{\mathcal{H}}(\tau, \nu)$ [13, 14]

$$y(t) = \int_{\tau} \int_{\nu} S_{\mathcal{H}}(\tau, \nu)x(t - \tau)e^{j2\pi\nu t}d\tau d\nu,$$  \hspace{1cm} (3)

where the received signal $y(t)$ is constructed by a continuous ensemble of rays based on the input signal $x(t)$ shifted in time by $\tau$ and in frequency by $\nu$ and attenuated by a complex factor as described by the spreading function $S_{\mathcal{H}}(\tau, \nu)$. We focus on underspread channels with spreading functions concentrated in time-frequency, e.g. their support area $\tau \times \nu$ is limited. The support of spreading functions for ideal, time dispersive, frequency dispersive and time-frequency dispersive channels are shown in Fig. 1.

A. The ideal channel

For an idealized channel characterized by $S_{\mathcal{H}}(\tau, \nu)\delta(\tau)\delta(\nu)$ any square-integrable signal $x(t)$ is an ‘eigenfunction’ as the input-output relation reduces to $y(t) = S_{\mathcal{H}}(0, 0) \cdot x(t)$. An orthogonal set of signals $s_k$ is usually preferred to achieve optimal detection in AWGN.

Fig. 1. Support of different underspread spreading functions a) ideal, b) time dispersive c) frequency dispersive and d) time-frequency dispersive with elliptical support (figure based on [14]).

B. Time dispersive LTI and frequency dispersive LFI channels

For an LTI time-dispersive system it is well-known that complex exponentials over time form a set of approximate eigenfunctions [1, 3]. Approximate because complex exponentials have infinite support and are therefore not integrable. An illustration of the approximate eigenfunctions and their time and frequency representations are shown in Fig. 2a. For an LFI channel the dual signals in time-frequency, the complex exponentials over frequency, form a set of approximate eigenfunctions [3]. The time-representation consists of time-shifted delta-pulses as shown in Fig. 2c. Instead of using non-physical complex exponentials and delta pulses, approximate signal sets can be used. For dominantly time dispersive channels the usage of windowed complex exponentials like Orthogonal Frequency Division Multiplexing (OFDM) is justified. Similarly, for dominantly frequency dispersive channels band-limited delta pulses can be used, which basically forms a set of signals which is the time-frequency dual of OFDM.

C. The time-frequency dispersive LTV channel

Special interest goes to LTV time-frequency dispersive channels. We assume a wide-sense stationary uncorrelated scattering (WSSUS) channel, being characterized by a zero-mean spreading function $E(S_{\mathcal{H}}) = 0$ and non-correlated scatters, i.e., $E(S_{\mathcal{H}}(\tau, \nu)S_{\mathcal{H}}^*(\tau', \nu')) = C_{\mathcal{H}}(\tau, \nu)\delta(\tau - \tau')\delta(\nu - \nu')$, where $C_{\mathcal{H}}$ represents the scattering function. A real-valued scatter function $C_0$ with elliptical symmetry in time-frequency (using parameters $\tau_0$ and $\nu_0$) is assumed:

$$C_{\mathcal{H}}(\tau, \nu) = C_0 \left( \frac{\tau}{\tau_0} \right)^2 + \left( \frac{\nu}{\nu_0} \right)^2.$$  \hspace{1cm} (4)

Kozek and Molisch [1] showed that (dilated) Hermite functions constitute the local optima to obtain minimum orthogonal distortion for this scattering function during propagation through a WSSUS channel. Minimum distortion signals for more general classes of scatter functions is a remaining problem. Despite, the formulation in (4) can approximate a large class of scattering functions and is intuitively appealing for Gaussian distributed delay and Doppler spread. The five lowest-order Hermite functions are shown in Fig. 2b. The Hermite functions are dilated, i.e., scaled in time, according to the time and frequency dispersion ratio of (4):

$$h_n \left( \frac{t}{\phi} \right) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \cdot H_n \left( \frac{t}{\phi} \right) \cdot e^{-\frac{t^2}{2\phi^2}}$$  \hspace{1cm} (5)

with $\phi = \sqrt{\frac{\tau_0}{2\pi\nu_0}}$, $H_n(y) = (-1)^n e^{y^2} d^n e^{-y^2}$. By setting $s_k(t) = h_n(t/\phi), n = k$ we construct the (complex) base-band signal of (1) based on a set of Hermite functions. The Hermite functions constitute a complete orthogonal set

$$\int_{-\infty}^{\infty} h_n \left( \frac{t}{\phi} \right) \cdot h_m \left( \frac{t}{\phi} \right) dt = \begin{cases} \phi & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$  \hspace{1cm} (6)
Hermite functions are also the eigenfunctions of the Fourier transform and the generalized unitary FrFT operator [5]

\[
\mathcal{F}^\alpha \{ h_n(t) \} (u) = \int_{-\infty}^{\infty} K_\alpha(t, u) h_n(t) dt = \lambda_n h_n(u)
\]

where \( K_\alpha \) is the kernel of the FrFT. Transformation of a signal by the FrFT leads to a signal representation over axis \( u \) making an angle \( \alpha \) with the time-axis. The FrFT actually rotates the signal presentation in time-frequency over an angle \( \alpha \) and is periodic with \( \alpha = 2\pi n, n \in \mathbb{N} \). When \( \alpha \) is equal to \( \pi/2 \) or \(-\pi/2 \) we obtain a counterclockwise or clockwise rotation in time-frequency by 90 degrees and equation (7) reduces to the unitary forward and inverse Fourier transform, respectively. A Hermite function of order \( n \) is an eigenfunction of the FrFT operator with an eigenvalue equal to complex-valued \( \lambda_n \).

III. SYNCHRONIZATION IN EITHER TIME OR FREQUENCY

Although approximate eigenfunctions of the channel operators preserve their shape during transmission easing equalization, that is not a guarantee for perfect signal recovery. Synchronization is important to achieve optimal matched filtering in AWGN. This section deals with synchronization as a two-step approach by separating time and frequency synchronization, while the next section deals with simultaneous time-frequency synchronization. Recall that the eigenfunctions of purely time dispersive and frequency dispersive channels are complex exponentials over time and frequency, respectively.

As a consequence of the stationarity of a set of time-varying complex exponentials, their orthogonality is maintained regardless of a time-offset. This fact is used by the cyclic extension/prefix in OFDM. The orthogonality is preserved if the cyclic extension is larger than the maximum time offset possible. Subsequent synchronization over frequency becomes a one-dimensional problem where the optimum is found by correlating the frequency presentations of the received signal and a template signal. A similar reasoning holds for eigenfunctions of LFI channels, frequency-varying complex exponentials. A cyclic extension eases frequency synchronization and a time-shift can be compensated by correlating the time-domain presentations of the received signal and a template signal.

Complex exponentials over time and frequency and their associated conjugate representations \( \delta(\omega) \) and \( \delta(t) \) are 'ideal' pulses for synchronization as these functions are very well-defined in frequency or time, respectively. However, a delta pulse is poorly concentrated in its conjugate domain and both \( \delta(\omega) \) and \( \delta(t) \) are non-physical (assume either non-causality or non-bandlimitedness). The Hermite functions - and especially the first Hermite function, the Gaussian pulse - are known to be optimally concentrated in time-frequency (e.g. [4]). The next...
section discusses a procedure to use the Gaussian pulse, or some other (non-stationary) signal, to achieve synchronization and subsequent matched filtering in time-frequency.

IV. SYNCHRONIZATION AND MATCHED FILTERING IN TIME-FREQUENCY

Hermite functions, as well as other non-stationary signals, are sensitive to both time and frequency offsets. When these offsets are not compensated, a loss of orthogonality among the different subcarriers \( s_k \) occurs. We define \( T_{\Delta \tau} \) to represent an operator translating a signal over time by a time-delay \( \Delta \tau \). Similarly, the operator \( M_{\Delta \nu} \) represents a shift over frequency by a frequency offset \( \Delta \nu \). Time-frequency shifts are represented by \( T_{\Delta \tau} M_{\Delta \nu} \) and \( M_{\Delta \nu} T_{\Delta \tau} \), but the operations are not commutative, i.e., \( T_{\Delta \tau} M_{\Delta \nu} s(t) = e^{-j2\pi \Delta \tau \Delta \nu} M_{\Delta \nu} T_{\Delta \tau} s(t) \) yielding a constant phase-shift related to the time-frequency shift [15]. We assume that the operation of time-translation follows the operation of frequency-translation, i.e., \( T_{\Delta \tau} M_{\Delta \nu} s(t) \). A quadrature matched filter allows for optimal time-frequency detection of a received signal \( f(t) \) in AWGN (e.g. [16]),

\[
\left|\int_{-\infty}^{\infty} s(t) \overline{f(t)} dt\right|^2 \leq \int_{-\infty}^{\infty} |s(t)|^2 dt \cdot \int_{-\infty}^{\infty} |f(t)|^2 dt \tag{8}
\]

where the upper bar indicates complex conjugation and the left-hand side represents the quadrature matched filter. When \( f(t) \) is matched to the known template function \( s(t) \), i.e., the received signal \( f(t) \) is linearly dependent on \( s(t) \) and both belong to the class of square-integrable functions, equality is met in (8). We generalize (8) to optimum detection along arbitrary angles in time-frequency. The Cauchy-Schwarz inequality for the fractional Fourier transformed signals \( f(t) \) and \( s(t) \) is given by

\[
\left|\int_{-\infty}^{\infty} \mathcal{F}^{\alpha}(s(t))(u) \overline{\mathcal{F}^{\alpha}(f(t))(u)} du\right|^2 \\
\leq \int_{-\infty}^{\infty} \left|\mathcal{F}^{\alpha}(s(t))(u)\right|^2 du \cdot \int_{-\infty}^{\infty} \left|\mathcal{F}^{\alpha}(f(t))(u)\right|^2 du. \tag{9}
\]

Based on (8) and (9), we make three remarks. First, if \( s(t) \) and \( f(t) \) are linearly dependent, then so are \( \mathcal{F}^{\alpha}(s(t))(u) \) and \( \mathcal{F}^{\alpha}(f(t))(u) \). Second, the norm of a signal is preserved under the unitary FrFT. Third, the FrFT is a unitary operator which preserves its inner-product, i.e., \( \langle \mathcal{F}^{\alpha}(s(t)), \mathcal{F}^{\alpha}(f(t)) \rangle = \langle s(t), f(t) \rangle \). Accordingly, it follows that if the left-hand side of (8) is an optimal detector, then the left-hand side of (9) can be regarded as a generalized optimal detector evaluating signals along an arbitrary angle \( \alpha \) in time-frequency. As the received signal of interest \( f(t) \) is translated in time-frequency by \( \Delta \tau \) and \( \Delta \nu \), we actually need to search over the time-frequency plane for the optimal match. The aim is to find the maximum value of the detector for the time-frequency shifted template signal \( \hat{s}(t) \) with the signal of interest \( f(t) \)

\[
|\mathcal{R}_{s,f}(\tau, \nu)|^2 = \int_{-\infty}^{\infty} \mathcal{F}^{\alpha}(T_{\Delta \tau} M_{\Delta \nu} s(t))(u) \overline{\mathcal{F}^{\alpha}(f(t))(u)} du \tag{10}
\]

Instead of searching over the time-frequency plane for different \( \tau \) and \( \nu \), we propose to formulate (10) as a one-dimensional function (similar to a convolution/correlation integral). For the time and frequency shift we define two functions solely dependent on \( \theta \). Substituting for the time-shift \( d(\theta) \) and frequency-shift \( m(\theta) \) in (10) gives \( R_{s,f}(\theta) \) as a function of \( \theta \)

\[
R_{s,f}(\theta) = \int_{-\infty}^{\infty} \mathcal{F}^{\alpha}(T_{d(\theta)} M_{m(\theta)} s(t))(u) \overline{\mathcal{F}^{\alpha}(f(t))(u)} du. \tag{11}
\]

V. SYNCHRONIZATION AND MATCHED FILTERING USING THE SUNFLOWER SPIRAL

The real-valued functions \( d(\theta) \) and \( m(\theta) \) can describe any trajectory in the time-frequency plane. It is intuitive to start at the expected time and frequency shift \( d(0) = E(\Delta \tau) \) and \( m(0) = E(\Delta \nu) \) and search according to the turns of a spiral. We assume that time and frequency offsets are zero-mean such that \( m(0) = 0 \) and \( d(0) = 0 \). We define the maximum normalized time-frequency shift as \( \Delta \tau_{\text{max}} = \Delta \tau_{\text{max}} / T_s = \Delta \nu_{\text{max}} / T_s \) where \( \Delta \tau_{\text{max}} \) and \( \Delta \nu_{\text{max}} \) correspond to the maximum shifts to be detected, which are (not necessarily) related to the symbol time. Although many trajectories can be used, we propose to use a uniform, but irregular distribution of \( M \) points over the (normalized) time-frequency area \( \pi r_{\text{max}}^2 \) such that each time-frequency offset within the search space can be found with similar accuracy. The sunflower spiral appears to satisfy the criterion and is shown in Fig. 3. The spiral lateral growth program of a sunflower was first studied by Vogel [17] and assumes a fixed amount of area per seed. The description of the spiral is given by

\[
r(\theta) = e^{\sqrt{\theta} \cos(2\pi \gamma \theta)} \tag{12}
\]

where \( \gamma \) is called the golden ratio, the angle \( 2\pi \gamma^2 \) (\( \approx 137.5^\circ \)) is called the Fibonacci angle and \( c \) is adapted to get a radius of \( r_{\text{max}} \). Note that the parameter \( c \) is essential to control the time-frequency resolution, i.e., the number of points per area. Substituting \( d(\theta) = R(\theta) \) and \( m(\theta) = \Delta(\theta) \) - the real and imaginary parts of \( r(\theta) \), respectively - in (11) gives

\[
R_{s,f}(\theta) = \int_{-\infty}^{\infty} \mathcal{F}^{\alpha}(T_{\Delta \tau} M_{\Delta \nu} s(t))(u) \overline{\mathcal{F}^{\alpha}(f(t))(u)} du. \tag{13}
\]

Time-shifts in discrete time require fractional delay filters for any non-integer multiple of the sample interval. To avoid fractional delay filters, we can apply the FRFT identity [18]

\[
\mathcal{F}^{\alpha} T_{\Delta \tau} M_{\Delta \nu} s(t) = e^{j\phi} T_{\Delta \tau} M_{\Delta \nu} \mathcal{F}^{\alpha} s(t) \tag{14}
\]

with

\[
\begin{bmatrix}
\Delta \tau \\
\Delta \nu
\end{bmatrix} =
\begin{bmatrix}
\cos(\alpha) & \sin(\alpha) \\
-\sin(\alpha) & \cos(\alpha)
\end{bmatrix} \cdot
\begin{bmatrix}
\Delta \tau \\
\Delta \nu
\end{bmatrix}
\]

\[
\phi = \frac{1}{2} \sin(\alpha) \left((\Delta \tau^2 - \Delta \nu^2) \cos(\alpha) + 2\Delta \tau \Delta \nu \sin(\alpha)\right).
\]
Applying this identity to (13) and choosing the fractional angle as \( \alpha = \theta' = 2\pi\gamma^2 \theta \) leads to an expression containing FrFTs, time-shifts, but no frequency-shifts

\[
R_{\hat{s},f}(\theta) = e^{j\theta_1} \int_{-\infty}^{\infty} T_{cv} \bar{\sigma} \text{FrFT}_{\theta}(s(t))(u) \text{FrFT}_{\theta}(f(t))(u) du \tag{15}
\]

with \( \theta' = 2\pi\gamma^2 \theta \).

Instead, as stated before, we prefer to eliminate the fractional delay filters to ease implementation and achieve a description with only (complex) multiplications. Therefore, we choose \( d(\theta) = -\Im (r(\theta)) \) and \( m(\theta) = \Re (r(\theta)) \) and \( \alpha = \theta' \), giving

\[
R_{\hat{s},f}(\theta) = e^{j\theta} \int_{-\infty}^{\infty} M_{cv} \bar{\sigma} \text{FrFT}_{\theta}(s(t))(u) \text{FrFT}_{\theta}(f(t))(u) du \tag{16}
\]

This is the description for the spiral correlator (according to the trajectory of the sunflower spiral), which can be calculated without time-shifts.

**A. Discretized matched filter**

A discrete form of (16) requires an \( N \)-point sampled signal \( \tilde{f} \) and template function \( \hat{s} \) based on the signals \( f(t) \) and \( s(t) \), respectively. The sampling frequency should be larger than two times the highest possible frequency of the frequency shifted signal \( f(t) \). Similarly, the interval to be sampled should be at least the length of the support of the time shifted signal \( f(t) \). The spiral correlation would involve a discrete operation mapping \( N \) input points to \( M \) inner products for each time-frequency translation defined by the spiral as shown in Fig. 3. Calculating the \( M \)-point spiral correlation for \( \tilde{f} \) according to (16) involves \( N \) times \( M \) complex multiplications. The discrete form reads \( \hat{R}_{\hat{s},f} = A \cdot \tilde{f} \), where \( A \) is an \( M \times N \) matrix with on each row a vector describing \( e^{-j\beta M_{\nu} \text{FrFT}(s(t))(u)} \) multiplied by the discrete \( N \times N \) FrFT matrix. A discrete FrFT has been proposed by Candan, Ozaktas et al. [19]. Note that the discrete FrFT should fulfill (14) in order for the discrete approximation to hold. The matrix \( A \) can be calculated once and stored in a memory.

Dependent on the application, the number of steps \( M \) can be significantly reduced by using a partial spiral correlation. Namely, the search for the time-frequency offset can be stopped when the value \( \hat{R}_{\hat{s},f}(\theta) \) exceeds some threshold value. Accordingly, only a fraction of the maximum \( M \times N \) complex multiplications needs to be calculated leading to a significant reduction in the number of required multiplications.

**VI. Simulations**

The proposed spiral correlation procedure is simulated to illustrate synchronization and matched filtering for a set of Hermite functions. The procedure is as follows. First, using a Gaussian pulse (the zeroth order Hermite function) the time-frequency offset is estimated where we have taken \( \Delta \tau/T_s = 0.40 \) and \( \Delta \nu \cdot T_s = -0.20 \). The spiral as shown in Fig. 3 is used with 256 points, \( r_{\max} = 0.8 \) and the time-frequency offset is illustrated by the asterisk. Synchronization using the Gaussian pulse and the spiral correlation method gives

\[
\theta_{\max} = \arg \max_0 \hat{R}_{\hat{h}_0,\hat{h}_0}(\theta) ^2 \tag{14},
\]

which has been marked by a pentagon in Fig. 4 (\( \tau = 0.3559 \), \( \nu = -0.1893 \)). Second, given the estimated time-frequency offset the signal-to-interference ratio (SIR) is analyzed on a set of time-frequency shifted Hermite functions. The direct consequence of a small mis-synchronization (due to a limited number of points \( M \)) is a loss of orthogonality leading to a lower SIR. This effect is shown by matched filtering of the fifth order Hermite function and recovering its signal out of a set of 20 Hermite functions. We formulate the SIR as a function of \( \theta \) to illustrate the importance of a descent time-frequency synchronization. The SIR is defined as the subcarrier energy recovered divided by

**Fig. 4.** Synchronization using the spiral correlator for the Gaussian pulse \( h_0 \) and a time-frequency shifted Gaussian pulse \( \hat{h}_0 \). The pentagon marker indicates the 'best match', the highest spiral correlation output (0.9995) for the time-frequency shift as shown in Fig. 3.
be calculated. To avoid fractional delay filters, the optimal detector has been generalized to detection along arbitrary angles in time-frequency. The SIR is illustrative for the loss of orthogonality among the subcarriers due to a time-frequency shift, and has been simulated for an ensemble of Hermite functions which were ‘matched’ after synchronization using the spiral correlation method. The number of points and the occupied time-frequency area of the spiral determine the synchronization accuracy and directly relate to the achievable SIR. The presented results can be used for signal detection, synchronization, matched filtering of (non-stationary) multi-carrier signals and may prove useful in other contexts as well.

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VII. DISCUSSION & CONCLUSION

Spiral correlation is a reformulation of the optimum detector for time-frequency shifted signals in AWGN. Applying the sunflower spiral gives an approximately uniform distribution of points over the time-frequency plane. Other spirals can be used as well with particular advantages. E.g. the logarithmic spiral provides a higher resolution close to the expected time-frequency offset $E(\Delta \tau)$ and $E(\Delta \nu)$ and gradually decreases the resolution. A two-step approach may also prove useful; first a course-grained spiral correlation and around the obtained estimate a fine-grained spiral correlation.

Directions have been pointed out to use a spiral correlation method on discrete-time signals. Instead of a full spiral correlation also a partial correlation can be performed until a sufficient ‘match’ has been found. Accordingly, only a fraction of the total number of multiplications needs to

![Fig. 5. Signal-to-interference ratios as a function of $\theta$. The SIR is a function of $\theta$ which follows the offsets in time-frequency according to the sunflower spiral (Fig. 3). The SIR is based on the signal recovery of the fifth Hermite function out of a set of 20 Hermite functions in total with a time-frequency offset of $\Delta \tau = 0.40$ and $\Delta \nu - T_s = -0.20$. The maximum SIR is found at the nearest spiral point - which was found by synchronization using the Gaussian pulse $h_0$ - and has been indicated by a pentagon marker. The SIR is directly related to the resolution in time-frequency determined by the parameters $r_{\text{max}}$ and $M$.](image)